ARML Sample Tryout Questions

A - 1

Let N be a three-digit base ten integer whose middle digit is 0. N is a multiple of 11 and the quotient $\frac{N}{11}$ equals the sum of the squares of the digits of N. Compute N.

A - 2

If the third and fourth terms in an arithmetic progression are increased by 2 and 7 respectively, then the first four terms form a geometric progression. Find all possible values of the fourth term of the arithmetic progression.

C - 1

Let P be the set of consecutive integers ≥ 90. (Specifically, $P = \{90, 91, 92, 93, 94, \ldots\}$). Determine the 253rd member of the set P that is not divisible by 2, 3, 4 or 8.

C - 2

A square is inscribed in a 90° sector of a circle so that two of its vertices are on the circle and the other two vertices are on the pair of perpendicular radii. If a side of the square is 6 meters long and the area of the region bounded by the entire circle is $k\pi$ square meters, find the value of k.

ARML Sample Tryout Questions

D - 1

How many integers from 1 to 100 inclusive have no repeated prime factors?

D - 2

The equation $\left(2\sin x-\sqrt{3}\right)^2+\left(2\cos x-1\right)^2+\left(\tan x-\sqrt{3}\right)^2=0$ has exactly 8 solutions in the range $0° < x \le M°$, where x and M are given in degrees. Find the smallest possible value of M.

F - 1

If $\sin t=\frac{4\sqrt{5}}{9}(0°<t<90°)$, compute the value of $\left(\log_3 \sin t\right)+\left(\log_3 \cot t\right)$.

F - 2

An n-gon has six times as many diagonals as sides.
Let p denote the number of diagonals in a polygon with $(n - 1)$ sides.
Let q denote the number of diagonals in a polygon with $(n + 1)$ sides.
Determine the number of different prime factors of the product pq.

A – 1

Let *N* be a three-digit base ten integer whose middle digit is 0. *N* is a multiple of 11 and the quotient $\frac{N}{11}$ equals the sum of the squares of the digits of *N*. Compute *N*.

Let the three-digit integer N be $x0y$. Then $x + y = 11$ and $\frac{100x+y}{11} = x^2 + y^2$.

Thus, $100x + (11 - x) = 99x + 11 = 11(9x + 1) = 11x^2 + 11(11 - x)^2 \rightarrow 9x + 1 = x^2 + (11 - x)^2$

$\rightarrow 2x^2 - 31x + 120 = (2x - 15)(x - 8) = 0 \rightarrow x = 8$ and $y = 3 \rightarrow N =$ **<u>803</u>**

Alternative:

Here is a list of the 3-digit integers divisible by 11 and the sum of the squares of their digits {209 (85), 308 (73) , 407 (65), 506 (61), 605 (61), 704 (65), **<u>803</u> (73)**, 902 (85)}

Only in the highlighted pair is the first number a multiple of the second.

A – 2

If the third and fourth terms in an arithmetic progression are increased by 2 and 7 respectively, then the first four terms form a geometric progression. Find <u>all</u> possible values of the fourth term of the arithmetic progression.

Let the terms of the A.P. be a, $a+d$, $a+2d$, $a+3d$ (the usual suspects)

Then the terms of the G.P. are:

$t_1 = a$, $t_2 = ar = a + d$, $t_3 = ar^2 = a + 2d + 2$ and $t_4 = ar^3 = a + 3d + 7$

$t_4 \rightarrow a(r^3 - 1) = 3d + 7 \quad t_2 \rightarrow a(r - 1) = d$

Thus, $\frac{t_4}{t_2} = \frac{a(r^3-1)}{a(r-1)} = \frac{a(r-1)(r^2+r+1)}{a(r-1)} = r^2 + r + 1 = \frac{3d+7}{d}$ (Eqtn #1)

$r = \frac{t_3}{t_2} \rightarrow r = \frac{a+2d+2}{a+d}$ (Eqtn #2)

$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} \rightarrow t_2^{\,2} = t_1 \cdot t_3 \rightarrow (a + d)^2 = a(a + 2d + 2) \rightarrow d^2 = 2a$ (Eqtn #3)

Substituting for a in equation #2,

$$r = \frac{\frac{d^2}{2} + 2d + 2}{\frac{d^2}{2} + d} = \frac{d^2 + 4d + 4}{d^2 + 2d} = \frac{(d+2)^2}{d(d+2)} = \frac{d+2}{d} \quad \text{(Eqtn \#4) or } d = -2$$

$d = -2 \rightarrow a = 2 \rightarrow$ AP: 2, 0, -2, -4 and the GP: 2, 0, 0, 3

However, the latter is not a GP and $d = -2$ is extraneous.

Substituting for r in #1, $\left(\frac{d+2}{d}\right)^2 + \frac{d+2}{d} + 1 = \frac{3d+7}{d} \rightarrow (d + 2)^2 + d(d + 2) + d^2 = d(3d + 7)$

$\rightarrow 3d^2 + 6d + 4 = 3d^2 + 7d \rightarrow d = 4$ and $a = 8 \rightarrow a + 3d =$ **<u>20</u>**

[AP: 8, 12, 16, 20 (with d = +4) and the GP: 8, 12, 18, 27 (with $r = 3/2$)]

A – 2 continued

The above algebraic blizzard could have been avoided if the original AP had been represented as $a - 2d$, $a - d$, a, $a + d$ to simplify the representation of the 3rd and 4th terms of the GP.

If the GP is $a - 2d$, $a - d$, $a + 2$, $a + d + 7$, then equating the quotients of consecutive terms, i.e. he multiplier r, and cross multiplying, we have
$(a - d)^2 = (a - 2d)(a + 2)$ and $(a + 2)^2 = (a + d + 7)(a - d)$.
$\rightarrow d^2 + 4d = 2a$ and $d^2 + 7d = 3a - 4$
Subtracting, $3d = a - 4$ or $a = 3d + 4$
Substituting, $d^2 + 4d = 2(3d + 4) \rightarrow d^2 - 2d - 8 = (d - 4)(d + 2) = 0 \rightarrow d = -2$ or 4
This produces the same answers as above with a lot less stress.
Don't always jump into a problem assuming that trying the usual suspects will shorten the trial.

C – 1

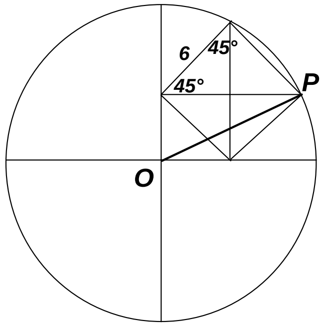
Let *P* be the set of consecutive integers $\geq$ 90. Specifically, *P* = {90, 91, 92, 93, 94, …}. Determine the 253rd member of the set *P* that is not divisible by 2, 3, 4 or 8.

Examining consecutive integers, every 2nd integer is a multiple of 2, every 3rd integer is a multiple of 3, etc.
Since the LCM of 2, 3, 4 and 8 is 24, we examine blocks of consecutive integers.
Every block of 24 integers contains exactly 8 integers that are not divisible by any of these 4 integers.

1 2 3 4 ***5*** 6 ***7*** 8 9 10 ***11*** 12 ***13*** 14 15 16 ***17*** 18 ***19*** 20 21 22 ***23*** 24 **|** ***25*** 26 27 28 ***29*** 30 ***31*** 32 33 34 ***35*** 36
Notice the repeating gaps of 3 and 1 (or vice versa) depending on where you start.
Since the quotient $\frac{253}{8}$ produces a quotient of 31 and a remainder of 5, we want the 5th integer in the 32nd block of 24 integers, starting with 90.
If 90 = **3**(24) + 18 starts the first block , then **34**(24) + 18 = 834 starts the 32nd block.
Since 91, 95, and 97 start the first block, the first integer is start +1 and the spacing is 3 – 1.
835 – 839 – 841 – 845 – **847**

C – 2

A square is inscribed in a 90° sector of a circle so that two of its vertices are on the circle and the other two vertices are on the pair of perpendicular radii. If a side of the square is 6 meters long and the area of the region bounded by the entire circle is *kπ* square meters, find the value of *k*.



Let the coordinates of O be (0, 0). Then the coordinates of P are $\left(6\sqrt{2}, 3\sqrt{2}\right)$ and the radius r of the circle is OP .
The area of circle O is πr^2. Thus, $k = 72 + 18 =$ **90**

D – 1
How many integers from 1 to 100 inclusive have no repeated prime factors?

We must eliminate any multiples of perfect squares from 1 to 100 inclusive.
Between 1 and 100 inclusive, the perfect squares are 4, 9, 25 and 49 and there are 25 multiples of 4, 11 multiples of 9, 4 multiples of 25 and 2 multiples of 49. The only integers in more than one multiple list will be multiples of 4·9 = 36 or 4·25 = 100.
The integers occurring in more than one multiple list are 36, 72 and 100.
Thus, we have 100 – (25 + 11 + 4 + 2) + (2 + 1) = **<u>61</u>**

D – 2
The equation $\left(2\sin x-\sqrt{3}\right)^2+\left(2\cos x-1\right)^2+\left(\tan x-\sqrt{3}\right)^2=0$ **has exactly 8 solutions in the range** $0° < x \leq M°$**, where** x **and** M **are given in degrees. Find the smallest possible value of** M**.**

Each term is nonnegative and, therefore, each term must be zero to produce a sum of 0.
Thus, the first term requires x to be coterminal with 60° or 120°.
The second term requires x to be coterminal with 60° or 300°.
The third term requires x to be coterminal with 60° or 240°.
The only common ground is $60 + 360n$.
If the first solution is 60 + 0·360, then the 8th solution is 60 + 7·360 = **<u>2580</u>**°.

F – 1
If $\sin t=\dfrac{4\sqrt{5}}{9}(0°<t<90°)$**, compute the value of** $\left(\log_3\sin t\right)+\left(\log_3\cot t\right)$**.**

Since $\sin^2 t + \cos^2 t = 1$ and t is in the first quadrant, we have $\cos t = \dfrac{1}{9}$.

$\left(\log_3\sin t\right)+\left(\log_3\cot t\right) = \log_3(\sin t\cdot\cot t) = \log_3\cos t = \log_3\left(3^{-2}\right) =$ **<u>-2</u>**

F - 2
An ***n*****-gon has six times as many diagonals as sides.**
Let p **denote the number of diagonals in a polygon with (**n **– 1) sides.**
Let q **denote the number of diagonals in a polygon with (**n **+ 1) sides.**
Determine the number of different prime factors of the product pq**.**

We require n such that $6n=\dfrac{n(n-3)}{2}$.
This quadratic equation has a solution of 15 (0 is extraneous).
Thus, a 14-gon has 77 diagonals and a 16-gon has 104 diagonals and $pq = 77(104) = 7\cdot 11\cdot 2^3\cdot 13$ → **<u>4</u>**